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Resonant wave interactions near a critical level in a stratified shear flow

By R. GRIMSHAW

Department of Mathematics, Monash University, Clayton VIC 3168, Australia

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Resonant interactions between internal gravity waves propagating in a stratified shear flow are considered for the case when the background density and shear flow vary slowly with respect to the waves. In Grimshaw (1988) triad resonances were considered, and interaction equations derived for the case when the resonance conditions are met only on certain space–time surfaces, being resonance sites. Here this analysis is extended to include higher-order resonances, with the aim of studying resonant wave interactions near a critical level. It is shown that a secondary resonant interaction between two incoming waves, in which two harmonic components of one incoming wave interact with a single harmonic component of another incoming wave, produces a reflected wave. This result is shown to agree with the study of Brown & Stewartson (1980, 1982*a*, *b*) who obtained this same result by a different approach.

1. Introduction

Resonant wave interactions in fluid flows have been intensively studied over the last two decades (see, for instance, Craik 1986). For the most part, attention has focused on the case when the resonance conditions are met globally. In this case the interacting waves are infinite periodic wavetrains, the interaction equations can be integrated in terms of elliptic functions, and the nature of the interaction is well understood. If, instead, the interacting waves are wave packets, the interaction equations can sometimes be solved by an inverse scattering transform algorithm and many interesting solutions have been found. However, situations can arise when the resonance conditions can only be met locally on certain space–time surfaces. This has been studied in a model context by Grimshaw (1987), and for triad interactions in stratified shear flows by Grimshaw (1988, hereafter denoted by (G)).

In this paper we extend the analysis of (G) to consider higher-order interactions in stratified shear flows for the case when the background stratification and shear flow vary slowly on length and time scales associated with the waves. Resonant triad interactions for internal gravity waves in the absence of any background shear flow have been studied by Ball (1964) for a two-layer fluid, and by Thorpe (1966) for continuous stratification. When a background shear flow is included, resonant triad interactions have been studied in layered fluids by Cairns (1979), Craik & Adam (1979), Tsutuhara (1984) and Tsutuhara & Hashimoto (1986), while Becker & Grimshaw (1993) have considered the case of continuous background stratified shear flows. However, in all these studies the resonance conditions are met globally, whereas in (G) the resonance conditions are only met locally on certain resonance sites, being space–time surfaces.

Our main aim in extending the analysis of (G) to higher-order resonances is to be able to study resonant wave interactions near a critical level. In a series of papers, Brown & Stewartson (1980, 1982*a*, *b*, hereafter denoted as (BS)) studied the nonlinear processes affecting internal gravity waves propagating towards a critical level. They showed that wave reflection and transmission were determined by a sequence of resonant wave interactions in which the higher harmonics of the main incoming wave played a crucial role in the interaction process. (BS) showed that the main incoming wave, denoted as a steady wave being the wave which defines the critical level, and another incoming wave, denoted as critical layer noise being a wave transient associated with the generation of the main wave, interact near the critical level to produce first a reflected wave, and then at a higher order, a transmitted wave. The analysis of (G), although confined to triad resonant interactions, was able to mimic this process by allowing the steady wave and the critical-layer noise to have different harmonic components. But to reproduce the (BS) scenario exactly it is necessary to allow the steady wave and the critical-layer noise to have the same harmonic components, and this in turn makes it necessary to extend the analysis of (G) to include higher-order resonances. This we do here by considering both primary and secondary resonant triads.

In §2 we describe a modulated wave packet propagating in a stratified shear flow. As well as describing the modulation of the amplitude of the primary harmonic, we also include a calculation of the second harmonic and mean flow components. In contrast to (G) which used a Lagrangian coordinate system, we here use an Eulerian coordinate system to facilitate comparison with (BS). Then in §3 we derive the interaction equations for a primary resonant triad, reproducing the result of (G), and also for secondary resonant triads, in which either a second harmonic of one wave interacts with the primary harmonic of two other waves (a second harmonic interaction), or a bound harmonic produced by the interaction of two primary harmonics interacts with one of these primary harmonics and a third wave (a bound harmonic interaction). In §4 we apply this general theory to study wave interactions near a critical level. For this purpose, we consider only two-dimensional flow configurations, assume that the background shear is linear and that the density stratification is uniform. For simplicity we also make a hydrostatic approximation here. Although the main purpose of these specializations is to facilitate a direct comparison with (BS), we emphasize that many analytical and numerical studies of critical layers have been carried out for this same special case. As in (BS) we consider two incoming waves, a steady wave and critical-layer noise, and then consider the possible primary and secondary resonant interactions. We show that there are no primary resonant triads, but a secondary resonance occurs which produces a reflected wave. The reflection coefficient of this wave is calculated and found to agree exactly with the result of (BS). This demonstrates that the nonlinear analysis of (BS) can be interpreted as resonant wave interactions in the sense of this paper.

2. Modulated waves

Let the basic flow consist of the shear flow $\boldsymbol{u}_0(z) = (u_0(z), v_0(z), 0)$ and the density field $\rho_0(z)$. Throughout we shall use non-dimensional variables based on a lengthscale h_1 (a typical wavelength), a timescale N_1^{-1} (where N_1 is a typical value of the Brunt–Väisälä frequency) and a pressure scale $\rho_1 g h_1$ (where ρ_1 is a typical value of the density). The non-dimensional Brunt–Väisälä frequency is $N(z)$, where

$$\frac{\partial \rho_0}{\partial z} = -\beta \rho_0 N^2. \tag{2.1}$$

Here $\beta = h_1 N_1^2 g^{-1}$ is the Boussinesq parameter. We shall assume that the fluid is non-diffusive and incompressible, so that the density ρ is given for all time by

$$\rho = \rho_0(z-\zeta), \tag{2.2}$$

where ζ is the vertical displacement of a fluid particle from its initial position. Thus

$$w = \frac{d\zeta}{dt} + \boldsymbol{u}\cdot\nabla\zeta, \tag{2.3a}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u}_0\cdot\nabla. \tag{2.3b}$$

Here $\boldsymbol{u}$ is the velocity field relative to the basic flow, and $w = \boldsymbol{u}\cdot\boldsymbol{k}$ is the vertical velocity component. The incompressibility condition is

$$\nabla\cdot\boldsymbol{u} = 0. \tag{2.4}$$

To complete the equations of motion we introduce a pressure perturbation q by

$$p = p_0(z) + \beta q, \tag{2.5}$$

where p is the pressure, and $p_0(z)$ is the basic pressure field. Then the momentum equation is

$$\rho\left\{\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\frac{\partial \boldsymbol{u}_0}{\partial z}\right\} + \nabla q + \frac{\boldsymbol{k}}{\beta}\{\rho - \rho_0(z)\} = \boldsymbol{0}. \tag{2.6}$$

Next we assume that the basic flow varies slowly with respect to the length and time scales of the wave field. Hence we introduce the slow variables

$$\boldsymbol{X} = \epsilon\boldsymbol{x}, \quad T = \epsilon t, \tag{2.7}$$

where ϵ is a small parameter. The basic flow is assumed to be a function of $Z = \epsilon z$, so that $\boldsymbol{u}_0 = \boldsymbol{u}_0(Z)$ and $\rho_0 = \rho_0(Z)$. Consequently from (2.2) $\rho = \rho_0(Z - \epsilon\zeta)$. Consistent with these hypotheses we assume that the Boussinesq parameter β is $O(\epsilon)$ and we put $\beta = \sigma\epsilon$. Then (2.1) becomes

$$\frac{\partial \rho_0}{\partial Z} = -\sigma\rho_0 N^2. \tag{2.8}$$

It follows that

$$\rho - \rho_0(Z) = \epsilon\sigma\rho_0 N^2\zeta + \sigma\epsilon^2 N_1, \tag{2.9a}$$

where

$$N_1 = -\tfrac{1}{2}(\rho_0 N^2)_Z\,\zeta^2 + O(\epsilon\zeta^3). \tag{2.9b}$$

Further, we now write the momentum equation (2.6) in the form

$$\rho_0(z)\left\{\frac{d\boldsymbol{u}}{dt} + \epsilon w\frac{\partial \boldsymbol{u}_0}{\partial Z}\right\} + \nabla q + \rho_0 N^2\zeta\boldsymbol{k} + \boldsymbol{M} = \boldsymbol{0}, \tag{2.10a}$$

where

$$\boldsymbol{M} = \rho_0(z)\,\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \epsilon\boldsymbol{k}N_1 + \sigma\epsilon(\rho_0 N^2\zeta + \epsilon N_1)\left(\frac{d\boldsymbol{u}}{dt} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \epsilon w\frac{\partial \boldsymbol{u}_0}{\partial z}\right). \tag{2.10b}$$

Eliminating q and $\boldsymbol{u}$ from the linear parts of (2.3a), (2.4) and (2.10a) in favour of ζ we get

$$\frac{\partial}{\partial z}\left(\rho_0\frac{d^2}{dt^2}\left(\frac{\partial\zeta}{\partial z}\right)\right) + \rho_0\left(\frac{d^2}{dt^2} + N^2\right)\nabla_H^2\,\zeta + M = 0, \tag{2.11a}$$

where

$$M = \nabla_H^2 M_V - \frac{\partial}{\partial z}(\nabla_H \cdot M_H) + \frac{d}{dt}\left(\rho_0 \nabla_H^2 I + \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial I}{\partial z}\right)\right) - \epsilon^2 \frac{\partial}{\partial Z}\left(\rho_0 \frac{\partial \boldsymbol{u}_0}{\partial Z}\right) \cdot \nabla_H I, \tag{2.11b}$$

and

$$I = \boldsymbol{u} \cdot \nabla \zeta. \tag{2.11c}$$

Here the subscripts H and V denote horizontal and vertical components respectively. In (2.11 *a*) M contains all the nonlinear terms.

Now the linear part of (2.11 *a*) is identical with the corresponding linear operator in (*G*). Hence we employ the same methodology although we note that the nonlinear terms are different since the present development is in Eulerian coordinates whereas that in (*G*) was based on Lagrangian coordinates. Thus (2.11 *a*) can be written in the form

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \boldsymbol{x}}; Z; \epsilon\right) + M = 0. \tag{2.12}$$

Here $L(p_0, \boldsymbol{p}; Z; \epsilon)$ is the linear operator defined by

$$L = L_0 + \epsilon L_1 \tag{2.13a}$$

where

$$L_0(p_0, \boldsymbol{p}; Z) = \rho_0\{(p_0 + \boldsymbol{u}_0 \cdot \boldsymbol{p}_H)^2 \boldsymbol{p}^2 + N^2 \boldsymbol{p}_H^2\}, \tag{2.13b}$$

and

$$L_1(p_0, \boldsymbol{p}; Z) = \frac{1}{2}\frac{\partial}{\partial Z}\left(\frac{\partial L_0}{\partial p_3}\right). \tag{2.13c}$$

Here $p_3 = \boldsymbol{p} \cdot \boldsymbol{k}$ is the vertical component of $\boldsymbol{p}$.

In the remainder of this section we seek an asymptotic description of a single modulated wave. Thus we put

$$\zeta = \alpha\{A(\boldsymbol{X}, T)\exp(i\theta) + *\} + \epsilon\alpha^2 \zeta^{(2)} + O(\epsilon^2 \alpha^3), \tag{2.14a}$$

where

$$\theta = \frac{1}{\epsilon}\Theta(\boldsymbol{X}, T). \tag{2.14b}$$

The wave is described by a slowly varying amplitude A and a rapidly varying phase θ. The local frequency ω and wavenumber $\boldsymbol{\kappa}$ are defined by

$$\omega = -\frac{\partial \Theta}{\partial T}, \quad \boldsymbol{\kappa} = \nabla \Theta. \tag{2.15}$$

Note that because an unmodulated wave is an exact solution of the fully nonlinear equations (e.g. Grimshaw 1974), we can insert a factor ϵ into the coefficient of the perturbation term $\zeta^{(2)}$. In (2.14*a*) α is an amplitude parameter, and in the next section will be regarded as small. However, for the time being, it can be finite and (2.14*a*) is essentially an expansion in ϵ. Clearly $\zeta^{(2)}$ will consist of a second harmonic term and a mean term, and so we anticipate that we may eventually put

$$\zeta^{(2)} = \{A^{(2)}\exp(2i\theta) + *\} + A^{(0)}. \tag{2.16}$$

Our main purpose here is the calculation of the second harmonic term $A^{(2)}$, although for completeness we shall also calculate the mean term $A^{(0)}$. However, this latter calculation requires a different procedure from that for the second harmonic, and hence is deferred until the end of this section.

As a preliminary to finding $\zeta^{(2)}$ we define the dispersion operator

$$D(\omega, \boldsymbol{\kappa}; Z) \equiv L_0(-\mathrm{i}\omega, \mathrm{i}\boldsymbol{\kappa}; Z), \tag{2.17a}$$

so that

$$D \equiv \rho_0\{\hat{\omega}^2\kappa^2 - N^2\kappa_H^2\}, \tag{2.17b}$$

where

$$\hat{\omega} = \omega - \boldsymbol{u}_0\cdot\boldsymbol{\kappa}_H, \quad \kappa = |\boldsymbol{\kappa}|, \quad \kappa_H = |\boldsymbol{\kappa}_H|. \tag{2.17c}$$

Here (2.17*b*) follows from (2.13*b*). Substituting (2.14*a*) into (2.12) we find that, at leading order,

$$D(\omega, \boldsymbol{\kappa}; Z) = 0 \tag{2.18}$$

and so, as expected, the modulated wave satisfies the local dispersion relation. At the next order, we obtain (see (G))

$$\epsilon\alpha^2 L_0\,\zeta^{(2)} + \epsilon\alpha\{\mathrm{i}D_1 A \exp(\mathrm{i}\theta) + *\} + M^{(2)} + \ldots = 0, \tag{2.19a}$$

where

$$D_1 A = \frac{\partial D}{\partial\omega}\frac{\partial A}{\partial T} - \frac{\partial D}{\partial\boldsymbol{\kappa}}\cdot\nabla A + \frac{1}{2}\left[\frac{\partial}{\partial T}\left(\frac{\partial D}{\partial\omega}\right) - \nabla\cdot\left(\frac{\partial D}{\partial\boldsymbol{\kappa}}\right)\right]A. \tag{2.19b}$$

Here and subsequently ∇ denotes the derivative with respect to X whenever the argument implies that this must be the case. In (2.19*a*) $M^{(2)}$ is the $\epsilon\alpha^2$ term in the nonlinear expression M, and ... denotes all higher-order terms. To avoid secularities $\zeta^{(2)}$ cannot contain any terms whose phase is θ since L_0 is then a null operator. Since it will transpire that $M^{(2)}$ contains no terms whose phase is θ, it follows that

$$D_1 A = 0. \tag{2.20}$$

Thus (2.18) and (2.20) determine the phase and amplitude respectively of the modulated wave. Further (2.20) is the wave action equation, and can be put in the form

$$\frac{\mathrm{D}J}{\mathrm{D}T} + \nabla\cdot(VJ) = 0, \tag{2.21a}$$

where

$$V = \nabla_\kappa\,\hat{\omega}, \tag{2.21b}$$

$$\kappa_H^2 J = \frac{\partial D}{\partial\omega}A^2 = 2\rho_0\,\hat{\omega}\kappa^2 A^2 \tag{2.21c}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}T} = \frac{\partial}{\partial T} + \boldsymbol{u}_0\cdot\nabla. \tag{2.21d}$$

Here V is the intrinsic group velocity, and J is the (complex) wave action density (see Grimshaw 1984).

The next task is to calculate $M^{(2)}$. First we note that, with ζ given by (2.14*a*),

$$\boldsymbol{u} = \alpha\boldsymbol{\eta}\{-\mathrm{i}\hat{\omega}A\exp(\mathrm{i}\theta) + *\} + \ldots, \tag{2.22a}$$

where

$$\boldsymbol{\eta} = \boldsymbol{k} - \frac{m\boldsymbol{\kappa}_H}{\kappa_H^2}, \quad m = \boldsymbol{\kappa}\cdot\boldsymbol{k}. \tag{2.22b}$$

Here the omitted terms contain the second-harmonic and mean terms. Note in particular that $\boldsymbol{\eta}\cdot\boldsymbol{\kappa} = 0$, and so, to leading order, the velocity field is perpendicular to the direction of phase propagation. At this stage, in calculating $M^{(2)}$ we will only collect the terms that contribute to the second harmonic $A^{(2)}$ in $\zeta^{(2)}$ (see (2.16)). The calculation of the mean term $A^{(0)}$ requires a different procedure, and will be left until the end of this section. First we consider I (2.11 c) and find that to leading order the second harmonic terms in I are given by

$$I^{(2)} = \epsilon\alpha^2\nabla\cdot(\hat{\omega}\boldsymbol{\eta})\{\mathrm{i}A^2\exp(2\mathrm{i}\theta)+*\}. \tag{2.23}$$

Similarly, it can be shown from (2.10 b) that to leading order the second-harmonic terms in $\boldsymbol{M}$ are given by

$$\boldsymbol{M}^{(2)} = \epsilon\alpha^2\{\hat{\omega}^2[\boldsymbol{\eta}(\nabla\cdot\boldsymbol{\eta})-(\boldsymbol{\eta}\cdot\nabla)\boldsymbol{\eta}]-\tfrac{1}{2}(\rho_0 N^2)_Z\,\boldsymbol{k}+\sigma\rho_0 N^2\hat{\omega}^2\boldsymbol{\eta}\}\{A^2\exp(2\mathrm{i}\theta)+*\}. \tag{2.24}$$

Hence, from (2.11 b) we find that the leading-order second-harmonic terms in M are given by

$$M^{(2)} = \epsilon\alpha^2 P^{(2)}(A^2\exp(2\mathrm{i}\theta)+*), \tag{2.25 a}$$

where
$$P^{(2)} = -2\kappa_H^2(\rho_0 N^2)_Z-6\rho_0 N^2\kappa^2\boldsymbol{\eta}\cdot\nabla\left(\frac{\kappa_H^2}{\kappa^2}\right)-12\rho_0 N^2\kappa_H^2\,\nabla\cdot\boldsymbol{\eta}. \tag{2.25 b}$$

Then, from (2.16) and (2.19 a) we find that, to leading order,

$$D(2\omega,2\boldsymbol{\kappa};Z)\,A^{(2)}+P^{(2)}A^2 = 0. \tag{2.26}$$

Further, from (2.17 b) and (2.18),

$$D(2\omega,2\kappa;Z) = 12\rho_0 N^2\kappa_H^2. \tag{2.27}$$

In particular, $D(2\omega,2\boldsymbol{\kappa};z)$ cannot be zero here and hence there are no second-harmonic resonances. Thus from (2.26) we have

$$A^{(2)} = \nu^{(2)}A^2, \tag{2.28 a}$$

where
$$\nu^{(2)} = -\frac{P^{(2)}}{12\rho_0 N^2\kappa_H^2}, \tag{2.28 b}$$

or
$$\nu^{(2)} = \nabla\cdot\boldsymbol{\eta}+\frac{\kappa^2}{2\kappa_H^2}\boldsymbol{\eta}\cdot\nabla\left(\frac{\kappa_H^2}{\kappa^2}\right)+\frac{(\rho_0 N^2)_Z}{6\rho_0 N^2}. \tag{2.28 c}$$

To complete the calculation of the first and second harmonics we must return to (2.3 a), (2.4) and (2.10 a), and calculate the counterpart of (2.14 a) for the velocity field $\boldsymbol{u}$ and the pressure q. Only the former will be needed in what follows and we put

$$\boldsymbol{u} = \alpha\{\boldsymbol{u}^{(1)}\exp(\mathrm{i}\theta)+\mathrm{c.c.}\}+\epsilon\alpha^2\{\boldsymbol{u}^{(2)}\exp(2\mathrm{i}\theta)+\mathrm{c.c.}\}+\alpha^2\boldsymbol{u}^{(0)}+\dots. \tag{2.29}$$

Here the first harmonic $\boldsymbol{u}^{(1)}$ is given to leading order by (2.22 a), while the mean term $\boldsymbol{u}^{(0)}$ will be calculated below. The second-harmonic term is given by, to leading order,

$$\boldsymbol{u}^{(2)} = \mathrm{i}\hat{\omega}A^2(-\mu^{(2)}\boldsymbol{\eta}+\sigma^{(2)}\boldsymbol{\kappa}\times\boldsymbol{k}), \tag{2.30 a}$$

where
$$\mu^{(2)} = \nu^{(2)}-\frac{(\rho_0 N^2)_Z}{3\rho_0 N^2}-\tfrac{1}{2}\sigma N^2, \tag{2.30 b}$$

and
$$\sigma^{(2)} = -\frac{\boldsymbol{\kappa}\times\boldsymbol{k}}{4\kappa_H^2}\cdot\nabla\left(\frac{\kappa^2}{\kappa_H^2}\right). \tag{2.30 c}$$

The mean flow component is best calculated by averaging the equations of motion with respect to the phase θ. The technique is well established, and for the present problem the results have been previously obtained by Grimshaw (1974). Hence we shall only give a brief outline here. First we define the averaging operator

$$\langle \dots \rangle = \frac{1}{2\pi}\int_0^{2\pi} (\dots)\, d\theta. \tag{2.31}$$

It follows that $\langle \boldsymbol{u} \rangle = \alpha^2 \boldsymbol{u}^{(0)}$, etc. Then we apply the averaging operator to (2.3*a*), (2.4) and (2.10*a*). We find that the average of the nonlinear terms $\boldsymbol{M}$ (2.10*b*) and I (2.11*c*) are given by, to leading order,

$$\langle \boldsymbol{M} \rangle = \epsilon\alpha^2\{\nabla\cdot(V\boldsymbol{\kappa}_H J) - \boldsymbol{k}\nabla\cdot\left(V\frac{\kappa_H^2 J}{m}\right) - \boldsymbol{k}(\rho_0 N^2)_Z |A|^2\} + \dots, \tag{2.32a}$$

and

$$\langle I \rangle = \epsilon^2\alpha^2 \nabla\cdot \boldsymbol{J}^{(0)} + \dots, \tag{2.32b}$$

where

$$\boldsymbol{J}^{(0)} = \frac{D}{DT}(|A|^2)\boldsymbol{\eta} + \nabla\cdot(\hat{\omega}\boldsymbol{\eta}|A|)\, 2|A|\frac{\kappa^2}{\kappa_H^2}\boldsymbol{\kappa}_H + \left\{\boldsymbol{\kappa}\times\boldsymbol{k}\cdot\left[\nabla\left(\frac{\hat{\omega}m|A|}{\kappa_H^2}\right) - 2|A|\,\boldsymbol{u}_{0Z}\right]\right\}\frac{|A|}{\kappa_H^2}\boldsymbol{\kappa}\times\boldsymbol{k}. \tag{2.32c}$$

Hence the equations for the mean flow are

$$\rho_0\left\{\frac{D\boldsymbol{u}^{(0)}}{DT} + \rho_0 w^{(0)}\frac{\partial \boldsymbol{u}_0}{\partial Z}\right\} + \rho_0 N^2 A^{(0)}\boldsymbol{k} + \nabla q^{(0)}$$
$$+\nabla\cdot(V\boldsymbol{\kappa}_H J) - \boldsymbol{k}\left\{\nabla\cdot\left(V\frac{\kappa_H^2}{m}J\right) + (\rho_0 N^2)_Z|A|^2\right\} = 0, \tag{2.33a}$$

$$\nabla\cdot\boldsymbol{u}^{(0)} = 0, \tag{2.33b}$$

and

$$w^{(0)} = \epsilon^2\left\{\frac{DA^{(0)}}{DT} + \nabla\cdot\boldsymbol{J}^{(0)}\right\}. \tag{2.33c}$$

Note in particular that the mean vertical velocity $\alpha^2 w^{(0)}$ is $O(\epsilon^2\alpha^2)$ and can usually be ignored. Also, the radiation stress forcing term in (2.33*a*) can be rewritten with the aid of the wave action equation (2.21*a*) as follows:

$$\nabla\cdot(V\boldsymbol{\kappa}_H J) = -\frac{D}{DT}(\boldsymbol{\kappa}_H J). \tag{2.34}$$

In what follows we are concerned only with the case when the wave packet varies only with respect to Z and T, and we can consequently assume that the mean flow also depends only on Z and T. In this situation it follows from (2.33*b*) that $w^{(0)}$ is zero, and then (2.33*c*) shows that

$$A^{(0)} = -\frac{\partial}{\partial Z}|A|^2. \tag{2.35}$$

Using (2.34), the horizontal component of the mean momentum equation (2.33*a*) shows that

$$\rho_0 \boldsymbol{u}_H^{(0)} = J\boldsymbol{\kappa}_H. \tag{2.36}$$

Finally, the vertical component of the mean momentum equation (2.33*a*) determines the mean pressure $q^{(0)}$.

3. Resonant interactions

We now seek an asymptotic solution of (2.11 *a*) which describes the interaction of two modulated waves to produce further modulated waves. Thus we put

$$\zeta = \sum_{r=0}^{1} \zeta_r + \delta \sum_{s=2}^{N} \zeta_s + \delta^2 \zeta^{(1)} + \dots. \tag{3.1}$$

Here each of ζ_0, ζ_1 is a single modulated wave, described by the analysis of the previous section, where now the wave variables are indexed by $r = 0, 1$. Thus, from (2.14 *a, b*) and (2.16),

$$\zeta_r = \alpha_r\{A_r \exp(\mathrm{i}\theta_r) + *\} + \epsilon\alpha_r^2\{A_r^{(2)} \exp(2\mathrm{i}\theta_r) + *\} + \epsilon\alpha_r^2 A_r^{(0)} + O(\epsilon^2\alpha_r^3). \tag{3.2}$$

Here $\omega_r, \boldsymbol{\kappa}_r$ satisfy the local dispersion relation (2.18), A_r satisfies the wave action equation (2.21 *a*), $A_r^{(2)}$ satisfies (2.28 *a*), and $A_r^{(0)}$ is determined from (2.35). Independently, for each $r = 0, 1$ they are asymptotically solutions of (2.11 *a*). However, their nonlinear interactions generate further modulated waves, which are described by the set $\zeta_s, s = 2, \dots, N$. The parameter δ is a generic ordering parameter. Then we put

$$\zeta_s = A_s \exp(\mathrm{i}\theta_s) + * \quad \text{for} \quad s = 2, \dots, N \tag{3.3}$$

and at leading order it is readily seen that the local dispersion relation (2.18) is satisfied for each $s = 2, \dots, N$ so that

$$D(\omega_s, \boldsymbol{\kappa}_s; Z) = 0. \tag{3.4}$$

At the next order we get (see (G) and (2.19 *a*)),

$$\delta^2 L_0 \zeta^{(1)} + \epsilon\delta \sum_{s=2}^{N} \{\mathrm{i}D_1 A_s \exp(\mathrm{i}\theta_s) + *\} + M^{(1)} = 0. \tag{3.5}$$

Here $D_1 A_s$ is defined by (2.19 *b*), and $M^{(1)}$ contains the leading-order nonlinear terms generated by the interaction of ζ_0 and ζ_1. There are also, of course, nonlinear terms generated by the interaction of $\zeta_r, r = 0, 1$ with $\zeta_s, s = 2, \dots, N$, or by interactions amongst the set ζ_s, but these are assumed here to be of higher order, and hence we will not need to consider them. To avoid secularities it is clear that $\zeta^{(1)}$ cannot contain any terms whose phase is θ_s (or indeed θ_0, θ_1) since L_0 is then a null operator (see 2.13 *b*) and (2.17 *a*)). It follows that

$$\mathrm{i}\epsilon\delta D_1 A_s + [M^{(1)} \exp(-\mathrm{i}\theta_s)] = 0, \tag{3.6}$$

where [...] indicates that this nonlinear term contributes only when a resonance occurs so that the phase of $M^{(1)}$ is θ_s. In the absence of such resonances, we see that $D_1 A_s = 0$, or A_s satisfies the wave action equation (see (2.21 *a–d*)),

$$\frac{\mathrm{D}J_s}{\mathrm{D}T} + \boldsymbol{\nabla}\cdot(V_s J_s) = 0, \tag{3.7a}$$

where

$$V_s = \boldsymbol{\nabla}_{\kappa_s} \hat{\omega}_s, \tag{3.7b}$$

and

$$\kappa_{Hs}^2 J_s = 2\rho_0 \omega_s \kappa_s^2 A_s^2. \tag{3.7c}$$

It remains to identify the leading-order resonances, and to determine the form of (3.6) when a resonance occurs. Here we identify three possibilities. The first occurs when the modes $r = 0, 1$ and $s = 2$, say, interact to form a resonant triad. This was the case discussed by (G) but we summarize the results here for convenience. The second occurs when the second harmonic of the mode $r = 0$ interacts with the first harmonic of the mode $r = 1$ to generate the mode $s = 2$, say. The third occurs when the first harmonics of the modes $r = 0, 1$ interact to produce a bound component, which then interacts with the first harmonic of the mode $r = 0$ to generate the mode $s = 2$ say.

(i) *Resonant triad interaction* This is the case discussed by (G) although the analysis there was in Lagrangian variables. Here we have repeated the analysis in the present Eulerian variables, and will summarize the outcome. Let us define

$$\theta_0+\theta_1+\theta_2 = \frac{1}{\epsilon}\chi(X, T), \tag{3.8a}$$

so that

$$\omega_0+\omega_1+\omega_2 = -\frac{\partial \chi}{\partial T}, \tag{3.8b}$$

and

$$\boldsymbol{\kappa}_0+\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2 = \boldsymbol{\nabla}\chi. \tag{3.8c}$$

Then a resonance occurs at those locations where $\partial\chi/\partial T$ and $\boldsymbol{\nabla}\chi$ vanish simultaneously. Here we are supposing that the modes $r = 0$ and 1 are interacting to produce mode 2, and hence we evaluate the term [...] in (3.6) for $s = 2$. Omitting details we find that

$$[M^{(1)}\exp(-\mathrm{i}\theta_2)] = -\mathrm{i}\alpha_0\,\alpha_1\,\rho_0\,N^2\kappa_{H2}^2\,\gamma A_0^*\,A_1^*\exp(-\mathrm{i}\chi/\epsilon), \tag{3.9a}$$

where

$$\gamma = \frac{m_2}{\kappa_2^2}(\boldsymbol{\eta}_0\cdot\boldsymbol{\kappa}_2)(\boldsymbol{\eta}_1\cdot\boldsymbol{\kappa}_2)+\ldots+\ldots. \tag{3.9b}$$

Here the omitted terms in (3.9*b*) are obtained by a cyclic interchange of the indices 0, 1, 2 and we recall that $\boldsymbol{\eta}$ and m are defined by (2.22*b*). Substitution of (3.9*a*) into (3.6) then gives the required equation for A_2. Note that (3.9*a, b*) corrects a minor misprint in (2.21 *a–c*) in (G).

The amplitude equation is thus (3.6) with the nonlinear term given by (3.9*a*) and D_1 defined by (2.19*b*). In what follows we do not need the full form of this equation, and it will be sufficient for us to treat it as an equation describing how the waves $r = 0, 1$ interact to produce the wave $s = 2$ near a resonance site, defined by (3.10) below. To simplify the form of (3.6) near a resonance, we follow (G) and assume that the resonance conditions

$$\omega_0+\omega_1+\omega_2 = 0, \quad \boldsymbol{\kappa}_0+\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2 = 0 \tag{3.10}$$

define a resonance surface $R(X, T) = 0$. On the resonance surface χ is a constant which we shall set to zero. Since χ_T and $\boldsymbol{\nabla}\chi$ vanish on $R = 0$, it follows that near $R = 0$ we may assume that $\chi \propto R^2$. Hence we rescale and put

$$R = \epsilon^{\frac{1}{2}}\tau, \quad \chi \approx \tfrac{1}{2}\epsilon S\tau^2, \tag{3.11}$$

where τ is a coordinate transverse to the resonance surface. Substituting these variables into (3.6), where we recall that D_1 is defined by (2.19*b*), and using (2.17*b*) and (3.9*a*) we get

$$\epsilon^{\frac{1}{2}}\delta\beta_2\frac{\partial A_2}{\partial \tau} = \alpha_0\,\alpha_1\,\gamma A_0^*\,A_1^*\exp(-\tfrac{1}{2}\mathrm{i}S\tau^2), \tag{3.12a}$$

where

$$\beta_2 = \frac{2}{\hat{\omega}_2}\left(\frac{\partial R}{\partial T}+V_2\cdot\boldsymbol{\nabla}R)\right). \tag{3.12b}$$

Here β_2 and γ are evaluated on the resonance surface $R = 0$. Equation (3.12*a*) now implies that the amplitude parameter $\delta = \alpha_0\,\alpha_1\,\epsilon^{-\frac{1}{2}}$ for this interaction, where we require that $\delta \ll 1$.

In general, there are two similar equations for A_0 and A_1 obtained by a cyclic interchange of the indices in (3.12*a*), and the three equations together define an exchange of energy between the three modes during the interaction. Here, however, we are taking the point of view that A_0 and A_1 are already specified, and (3.12*a*) describes the generation of A_2. Strictly, there are changes in the amplitudes of A_0 and A_1 during the resonant interaction, but these have magnitudes $\alpha_1\,\delta/\epsilon^{\frac{1}{2}}\alpha_0$ and $\alpha_0\,\delta/\epsilon^{\frac{1}{2}}\alpha_1$ respectively, and with $\delta = \alpha_0\,\alpha_1\,\epsilon^{-\frac{1}{2}}$ for this interaction, can be ignored if $\alpha_1^2 \ll \epsilon$ and $\alpha_0^2 \ll \epsilon$ respectively. Hence, in (3.12*a*) we suppose that A_0 and A_1 are constant (i.e. independent of the local variable τ), and so the solution is

$$A_2 = \frac{\gamma}{\beta_2} A_0^* A_1^* \int_{-\infty}^{\tau} \exp\left(-\tfrac{1}{2}\mathrm{i}S\tau'^2\right) \mathrm{d}\tau'. \tag{3.13}$$

Here, we are assuming that $A_2 \to 0$ as $\tau \to -\infty$ which will be the case if the sense of τ increasing is chosen to be consistent with causality. Then, as $\tau \to \infty$ corresponding to the generation of A_2, we get from (3.13),

$$A_2 \to \frac{\gamma}{\beta_2} A_0^* A_1^* \left(\frac{2\pi}{|S|}\right)^{\frac{1}{2}} \exp\left(-\frac{\mathrm{i}\pi}{4}(\mathrm{sign}\, S)\right). \tag{3.14}$$

This is the amplitude of A_2 generated by the interaction of A_0 and A_1, and is the main result of this subsection. Away from the resonance site, the amplitude is determined by (3.7*a*) (with $s = 2$), and (3.14) is effectively an initial condition for this equation.

(ii) *Second harmonic interaction* The description of this interaction is one of the two main purposes of this paper. We suppose that the second harmonic of the mode $r = 0$ interacts with the first harmonic of the mode $r = 1$ to generate the mode $s = 2$. Hence we define

$$2\theta_0 + \theta_1 + \theta_2 = \frac{1}{\epsilon}\hat{\chi}(X, T), \tag{3.15a}$$

so that

$$2\omega_0 + \omega_1 + \omega_2 = -\frac{\partial\hat{\chi}}{\partial T}, \tag{3.15b}$$

and

$$2\boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 = \nabla\hat{\chi}. \tag{3.15c}$$

Then a resonance occurs at those locations where $\partial\hat{\chi}/\partial T$ and $\nabla\hat{\chi}$ vanish simultaneously. Here we are supposing that the modes $r = 0$ and 1 are interacting to produce mode 2 and hence we evaluate the term [...] in (3.6) for $s = 2$ to get

$$[M^{(1)} \exp(-\mathrm{i}\theta_2)] = -\mathrm{i}\epsilon\alpha_0^2\alpha_1\rho_0 N^2\kappa_{H2}^2\gamma^{(2)} A_0^{*2} A_1^* \exp(-\mathrm{i}\chi/\epsilon), \tag{3.16a}$$

where

$$\gamma^{(2)} = \frac{m_0}{2\kappa_0^2}(\boldsymbol{\kappa}_0\cdot\boldsymbol{\eta}_1)(\boldsymbol{\kappa}_0\cdot\boldsymbol{\eta}_2)\,\hat{\nu}_0^{(2)} + \frac{m_1}{\kappa_1^2}(\boldsymbol{\kappa}_0\cdot\boldsymbol{\eta}_2)(\boldsymbol{\kappa}_2\cdot\hat{\boldsymbol{u}}_0^{(2)}) + \frac{m_2}{\kappa_2^2}(\boldsymbol{\kappa}_0\cdot\boldsymbol{\eta}_1)(\boldsymbol{\kappa}_1\cdot\hat{\boldsymbol{u}}_0^{(2)}), \tag{3.16b}$$

$$\hat{\nu}^{(2)} = -\nabla\cdot\boldsymbol{\eta} + \frac{\kappa^2}{2\kappa_H^2}\,\boldsymbol{\eta}\cdot\nabla\left(\frac{\kappa_H^2}{\kappa^2}\right) - \frac{\kappa_H^2}{m^2}\left(\frac{m^2}{\kappa_H^2}\right)_z + \frac{5\kappa^2}{2m^2}\boldsymbol{\eta}\cdot\nabla\left(\frac{m^2}{\kappa^2}\right) + \frac{5}{3}\frac{(\rho_0 N^2)_Z}{\rho_0 N^2}, \tag{3.16c}$$

$$\hat{\boldsymbol{u}}^{(2)} = \hat{\mu}^{(2)}\boldsymbol{\eta} + \sigma^{(2)}\boldsymbol{\kappa}\times\boldsymbol{k} + \hat{\sigma}^{(2)}\boldsymbol{\kappa}, \tag{3.16d}$$

$$\hat{\mu}^{(2)} = \frac{\kappa^2}{2\kappa_H^2}\boldsymbol{\eta}\cdot\nabla\left(\frac{\kappa_H^2}{\kappa^2}\right) + \frac{1}{3}\frac{(\rho_0 N^2)_Z}{\rho_0 N^2}, \tag{3.16e}$$

and

$$\hat{\sigma}^{(2)} = \frac{\kappa^2}{2m\kappa_H^2}\boldsymbol{\eta}\cdot\nabla\left(\frac{\kappa_H^2}{\kappa^2}\right), \tag{3.16f}$$

and we recall that $\sigma^{(2)}$ is defined by (2.30*c*). Here the coefficients $\nu_0^{(2)}$ and $\hat{\boldsymbol{u}}_0^{(2)}$ in (3.16*b*) are given by (3.16*c*, *d*) evaluated for the mode $r = 0$. The coefficients $\hat{\nu}^{(2)}$ and $\hat{\boldsymbol{u}}^{(2)}$ are, of course, determined from $\nu^{(2)}$ (2.28*c*) and $\boldsymbol{u}^{(2)}$ (2.30*a*), but the derivation is very lengthy and we shall omit the details. Note that $\gamma^{(2)}$ is symmetric with respect to the indices 1 and 2. Substitution of (3.16*a*) into (3.6) now gives the required equation for A_2.

As in case (i) above the amplitude equation (3.6) is regarded as an equation describing how the waves $r = 0, 1$ interact to produce the wave $s = 2$ near a resonance site, now defined by (3.17) below. To simplify the form of (3.6) near a resonance, we follow the same procedure as for case (i) and assume that the resonance conditions

$$2\omega_0 + \omega_1 + \omega_2 = 0, \quad 2\boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 = 0 \tag{3.17}$$

define a resonance surface $\hat{R}(X, T) = 0$. On this resonance surface $\hat{\chi}$ is a constant which we shall set to zero. Since $\hat{\chi}_T$ and $\nabla\hat{\chi}$ vanish on $\hat{R} = 0$, we assume that $\hat{\chi} \propto \hat{R}^2$, and so rescale so that

$$\hat{R} = \epsilon^{\frac{1}{2}}\tau, \quad \hat{\chi} \approx \tfrac{1}{2}\epsilon\hat{S}\tau^2. \tag{3.18}$$

Substituting these variables into (3.6) we now get

$$\epsilon^{\frac{1}{2}}\delta\hat{\beta}_2 \frac{\partial A_2}{\partial \tau} = \epsilon\alpha_0^2\alpha_1\gamma^{(2)}A_0^{*2}A_1^* \exp(-\tfrac{1}{2}\mathrm{i}\hat{S}\tau^2), \tag{3.19a}$$

where

$$\hat{\beta}_2 = \frac{2}{\hat{\omega}_2}\left(\frac{\partial \hat{R}}{\partial T} + V_2 \cdot \nabla \hat{R}\right). \tag{3.19b}$$

Here $\hat{\beta}_1$ and $\gamma^{(2)}$ are defined on the resonance surface $\hat{R} = 0$. Equation (3.19*a*) now implies that the amplitude parameter $\delta = \alpha_0^2\alpha_1\epsilon^{\frac{1}{2}}$ for this interaction.

There is a similar equation for A_1 obtained by an interchange of the indices 1 and 2 in (3.19*a*), and the two equations together define an exchange of energy between these two modes. Here, however, as in case (i) we are taking the point of view that A_0 and A_1 are already specified, and so (3.19*a*) determines the generation of A_2. Strictly there are changes in the amplitudes of A_0 and A_1 during the interaction, but these have magnitudes $\alpha_1\delta/\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{1}{2}}\alpha_0^2\delta/\alpha_1$ respectively, and with $\delta = \alpha_0^2\alpha_1\epsilon^{\frac{1}{2}}$ for this interaction, can be ignored if $\alpha_0\alpha_1 \ll 1$ and $\epsilon\alpha_0^4 \ll 1$. Hence, in (3.19*a*) we can suppose that A_0 and A_1 are constant, and so the solution is

$$A_2 = \frac{\gamma^{(2)}}{\hat{\beta}_2} A_0^{*2} A_1^* \int_{-\infty}^{\tau} \exp(-\tfrac{1}{2}\mathrm{i}\hat{S}\tau'^2)\,\mathrm{d}\tau', \tag{3.20}$$

where we are assuming that $A_2 \to 0$ as $\tau \to -\infty$. Then, as $\tau \to \infty$ we obtain the following expression for the generation of A_2:

$$A_2 \to \frac{\gamma^{(2)}}{\hat{\beta}_2} A_0^{*2} A_1^* \left(\frac{2\pi}{|\hat{S}|}\right)^{\frac{1}{2}} \exp\left(-\frac{\mathrm{i}\pi}{4}(\operatorname{sign}\hat{S})\right). \tag{3.21}$$

This is the amplitude of A_2 generated by the interaction of the second harmonic of the mode $r = 0$ with the first harmonic of the mode $r = 1$, and is the main result of this subsection. Away from this resonance site, as in case (i), the amplitude is determined by (3.7*a*) (with $s = 2$), and (3.21) is effectively an initial condition for this equation.

(iii) *Bound harmonic interaction* The description of this interaction is the second main purpose of this paper. We observe that the first harmonics of the modes $r = 0, 1$ can interact to produce a bound component with phase $(\theta_0 + \theta_1)$, where we are assuming here that there are no triad resonances in the vicinity. Then this bound component interacts with the first harmonic of the mode $r = 0$ to generate the mode $s = 2$. The resonance conditions are the same as those for case (ii) and hence given by (3.17).

To calculate the amplitude A_2 for this interaction we must first modify the expansion (3.1) to include the interaction which produces the bound harmonic. Thus we replace (3.1) with

$$\zeta = \sum_{r=0}^{1} \zeta_r + \zeta_{01} + \delta \sum_{s=2}^{N} \zeta_s + \delta^2 \zeta^{(1)} + \dots . \tag{3.22}$$

Here the bound harmonic is denoted by ζ_{01} and given by

$$\zeta_{01} = \alpha_0 \alpha_1 \{ \mathrm{i} \nu^{(0)} A_0 A_1 \exp(\mathrm{i}[\theta_0 + \theta_1]) + * \}. \tag{3.23}$$

The coefficient $\nu^{(0)}$ is calculated in a similar way to that for the second-harmonic coefficient $\nu^{(2)}$ (2.17), or indeed to that for the coefficient γ (3.9*b*) in the triad interaction. Thus we obtain

$$D(\omega_0 + \omega_1, \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1; Z) \zeta_{01} + M^{(10)} = 0, \tag{3.24}$$

where $M^{(10)}$ is the leading-order term in the nonlinear term M arising from the direct interaction of the first harmonics. Note that since we are assuming here that there are no triad resonances, we can assume that $D(\omega_0 + \omega_1, \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1; Z)$ is not zero. Then (3.24) provides the means for calculating $\nu^{(0)}$. We omit details and find that

$$D(\omega_0 + \omega_1, \boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1; Z) \frac{\nu^{(0)}}{\rho_0} = -|\boldsymbol{\kappa}_0 + \boldsymbol{\kappa}_1|^2 \{ \hat{\omega}_0^2 (\boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1) + \hat{\omega}_1^2 (\boldsymbol{\eta}_1 \cdot \boldsymbol{\kappa}_0) \} - 2\hat{\omega}_0 \hat{\omega}_1 (m_0 + m_1)(\boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1)(\boldsymbol{\eta}_1 \cdot \boldsymbol{\kappa}_0). \tag{3.25}$$

Next we assume that ζ_{01} interacts resonantly with ζ_0 to generate the mode ζ_2. As in case (ii) we define $\hat{\chi}$ by (3.14*a–c*) and then resonance occurs at those locations where $\partial \hat{\chi} / \partial T$ and $\nabla \hat{\chi}$ vanish simultaneously, or where the resonance conditions (3.17) hold. The remaining analysis is similar in form to that described in case (ii). We let (3.17) define a resonance surface $\hat{R}(\boldsymbol{X}, T) = 0$, then introduce τ by (3.18) and finally obtain from (3.6) the amplitude equation

$$\epsilon^{\frac{1}{2}} \delta \hat{\beta}_2 \frac{\partial A_2}{\partial \tau} = \alpha_0^2 \alpha_1 \gamma^{(0)} A_0^{*2} A_1^* \exp(-\tfrac{1}{2}\mathrm{i} \hat{S} \tau^2), \tag{3.26a}$$

where

$$\gamma^{(0)} = \frac{\mathrm{i}\hat{\nu}^{(0)}}{\hat{\omega}_2^2} \left\{ m_0 (\hat{\omega}_2 - \hat{\omega}_0) - \hat{\omega}_0 \frac{(m_0 m_2 + \kappa_{H2}^2)}{\kappa_2^2} \boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1 \right\} - \frac{\mathrm{i}\hat{\nu}^{(0)}}{\hat{\omega}_2^2} \frac{(m_0 + m_2)}{|\boldsymbol{\kappa}_{H0} + \boldsymbol{\kappa}_{H1}|^2} \left\{ \boldsymbol{\kappa}_{H0} \cdot (\boldsymbol{\kappa}_{H0} + \boldsymbol{\kappa}_{H1})(\hat{\omega}_2 - \hat{\omega}_0) + \frac{\hat{\omega}_0 m_2}{\kappa_2^2} (\kappa_{H0}^2 - \kappa_{H2}^2) \boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1 \right\} + \frac{\mathrm{i}\hat{\omega}_0}{\hat{\omega}_2} \boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1 \nu^{(0)} + \mathrm{i}\sigma^{(0)}, \tag{3.26b}$$

$$\hat{\nu}^{(0)} = (\hat{\omega}_0 + \hat{\omega}_1) \nu^{(0)} + \hat{\omega}_0 (\boldsymbol{\eta}_0 \cdot \boldsymbol{\kappa}_1) + \hat{\omega}_1 (\boldsymbol{\eta}_1 \cdot \boldsymbol{\kappa}_0), \tag{3.26c}$$

and

$$\sigma^{(0)} = \frac{(\boldsymbol{k} \cdot \boldsymbol{\kappa}_{H0} \times \boldsymbol{\kappa}_{H1})^2}{|\boldsymbol{\kappa}_{H0} + \boldsymbol{\kappa}_{H1}|^2} \left\{ \frac{m_1^2}{\kappa_{H1}^2} - \frac{m_0^2}{\kappa_{H0}^2} \right\} \left\{ \frac{\hat{\omega}_0 \hat{\omega}_1}{\hat{\omega}_2^2} + \frac{\hat{\omega}_1 (\kappa_{H2}^2 \kappa_{H0}^2 + m_2 m_0 \boldsymbol{\kappa}_{H2} \cdot \boldsymbol{\kappa}_{H0})}{(\hat{\omega}_1 + \hat{\omega}_0) \kappa_{H2}^2 \kappa_0^2} \right\}. \tag{3.26d}$$

Note that unlike its counterpart $\gamma^{(2)}$ in (3.19*a*) the coefficient $\gamma^{(0)}$ is not symmetric in the indices 1 and 2 owing to the different roles played by the modes 1 and 2 in this interaction. Equation (3.26*a*) implies that the amplitude parameter $\delta = \alpha_0^2 \alpha_1 \epsilon^{-\frac{1}{2}}$ for this interaction, and hence we require that $\alpha_0^2 \alpha_1 \ll \epsilon^{\frac{1}{2}}$. Significantly the amplitude parameter for this bound interaction is larger by a factor ϵ^{-1} than the corresponding amplitude parameter for the second-harmonic interaction.

The remaining analysis is similar to that following (3.19*a*) for the second harmonic interaction. We assume that A_0 and A_1 remain constant during the interaction which requires that $\alpha_0 \alpha_1 \ll \epsilon$ and that $\alpha_0^4 \ll 1$. Then the solution of (3.26*a*) is

$$A_2 = \frac{\gamma^{(0)}}{\hat{\beta}_2} A_0^{*2} A_1^* \int_{-\infty}^{\tau} \exp(-\tfrac{1}{2}\mathrm{i}\hat{S}\tau'^2)\,\mathrm{d}\tau', \tag{3.27}$$

where we are again assuming that $A_2 \to 0$ as $\tau \to -\infty$. Then as $\tau \to \infty$ we get

$$A_2 \to \frac{\gamma^{(0)}}{\hat{\beta}_2} A_0^* A_1^* \left(\frac{2\pi}{\hat{S}}\right)^{\frac{1}{2}} \exp\left(-\frac{\mathrm{i}\pi}{4}(\operatorname{sign}\hat{S})\right), \tag{3.28}$$

which is the amplitude of A_2 generated by this bound interaction. Away from the resonance site the amplitude is determined by (3.7*a*) (with $s = 2$), and (3.28) is effectively an initial condition for this equation.

4. Wave interactions near a critical level

The analysis of the previous section has identified a scenario in which wave triads interact in the vicinity of certain resonance surfaces. We envisage a sequence of such interactions in each of which a pair of waves interacts to produce a third wave, determined either by (3.10) for a primary resonant triad, or by (3.17) for a secondary resonant triad. During each such interaction the amplitude of the third wave produced is given by (3.14), (3.21) or (3.28) respectively. Outside the resonance surfaces the amplitude of each wave is determined by the appropriate wave action equation (3.7*a–c*). To illustrate this process we choose the particular example of waves approaching a critical level. This choice is motivated by the series of papers by Brown & Stewartson (BS) on the nonlinear processes affecting internal gravity waves near a critical level. They showed that wave reflection and transmission were determined by a hierarchy of wave interactions. Importantly for the present work they showed that the second harmonic of the main incoming wave plays a major role in initiating this process. In (G) we modelled this process using only primary resonant triad interactions (i.e. satisfying (3.10)). Here we follow the work of (BS) more closely, and imitate the process with a secondary resonant triad. Further, here we shall also keep track of the wave amplitudes.

4.1. *Wave classification*

To model a critical level, placed at $Z = 0$, we choose the Brunt–Väisälä frequency N to be constant, and put $\boldsymbol{u}_0 = (Z, 0, 0)$. The small parameter ϵ is a measure of $Ri^{-\frac{1}{2}}$ where Ri is the Richardson number of the basic flow at the critical level. We assume that $\boldsymbol{\kappa}_H$ (the horizontal component of $\boldsymbol{\kappa}$) is constant and parallel to $\boldsymbol{u}_0$, and to simplify the analysis we also assume that $m^2 \gg \kappa_H^2$, where m is the vertical wavenumber component. This is an appropriate assumption for waves near a critical level. Then the dispersion relation (2.18) has the approximate solutions

$$\hat{\omega} = \omega - kz = \mp \frac{N|k|}{m}, \tag{4.1}$$

where $\boldsymbol{\kappa} = (k, 0, m)$ and we recall that $\kappa_H = |k|$ is a constant. The alternate signs refer to waves whose vertical group velocity is positive (negative). Equation (4.1) is a partial differential equation for the phase Θ, where we recall from (2.15) that $\omega = -\Theta_T$ and $m = \Theta_z$. It is sufficient for our purposes to generate the following family of solutions of (4.1). We put $k = nK$ where n is an integer, $K > 0$ and let

$$\Theta = sN \ln T + kX + Nf(\eta) + E, \tag{4.2a}$$

where

$$\eta = KZT/N, \tag{4.2b}$$

s is an integer, and E is a disposable constant. Hence

$$\omega = -\frac{N}{T}(s + \eta f'(\eta)), \quad m = KTf'(\eta). \tag{4.3}$$

Substituting these expressions into (4.1) we get

$$\eta f'^2 + (n\eta + s) f' \mp |n| = 0. \tag{4.4}$$

The particular choice (4.2a, b) is motivated by the analysis of (BS) who found waves of this form in their study of nonlinear interactions near a critical level. (G) gives a comprehensive discussion of the solutions of (4.4), but for convenience we shall summarize that here. For each integer pair (n, s) there are two solutions of (4.3) with positive group velocity and two solutions with negative group velocity. We shall denote each solution as $\{n, s, i, \pm\}$ where $i = 1, 2$ refers to the two solution branche of (4.3) and $\pm$ refers to positive (negative) group velocity.

The trajectory of each wave is found by integrating the equation

$$\frac{dZ}{dT} = W = \pm\frac{N|k|}{m^2}, \tag{4.5}$$

where W is the vertical group velocity. Away from resonances the amplitudes are determined from the wave action equation (2.21a) which reduces here to

$$J_T + (WJ)_Z = 0, \tag{4.6a}$$

with

$$J = \text{constant} \times mA^2. \tag{4.6b}$$

It will be shown below that the resonance surfaces are level surfaces of η, and hence we may put $R, \hat{R} = (\eta + \text{constant})$. It then follows from (3.12b) and (3.19b) that

$$\beta_2, \hat{\beta}_2 = -\frac{2}{Nf'}\left(1 \pm \frac{\eta f'^2}{|n|}\right). \tag{4.7}$$

First, consider the solutions $\{n, \operatorname{sign} n, i, -\}$ which are given by

$$f' = -(\operatorname{sign} n)\,\eta^{-1}, \; -n \tag{4.8}$$

for $i = 1, 2$ respectively. The corresponding frequency and vertical wavenumber components are

$$\omega = 0, kZ - (\operatorname{sign} n)\frac{N}{T}, \tag{4.9a}$$

and

$$m = -(\operatorname{sign} n)\frac{N}{Z}, \; -kT. \tag{4.9b}$$

The first of these, $i = 1$, corresponds to a steady wave propagating downwards, and is just the steady wave analysed by Booker & Bretherton (1967), and found by them to undergo critical-layer absorption in the linear theory. Its trajectory is given by

$$\frac{KZ}{N} = (|n|\, T + \text{constant})^{-1}. \tag{4.10}$$

Thus, in $Z \to 0$ it represents a wave propagating towards the critical level at $Z = 0$ as $T \to \infty$, while in $Z < 0$ it represents a wave propagating away from the critical level with $Z \to -\infty$ in finite time. If it is assumed that the wave is generated far above the critical level, then its wavefront is given by $|n|\, \eta = 1$, and it occupies the region $|n|\, \eta \geqslant 1$. The second solution, $i = 2$, in (4.8) corresponds to critical-layer noise in the terminology of (BS) and is a transient associated with the start-up process. Its trajectory is

$$\frac{KZ}{N} = \frac{1}{|n|\, T} + \text{constant}. \tag{4.11}$$

Thus it represents a wave which propagates down to some finite level of Z as $T \to \infty$. Away from resonances the amplitudes are found from the wave action equation (4.6*a*). For the boundary condition used by (BS) at the level where the waves are generated, the amplitudes are proportional to $Z^{-\frac{1}{2}}$ and $T^{\frac{1}{2}}\, (|n|\, \eta - 1)^{-1}$ for $i = 1, 2$ respectively. The steady wave ($i = 1$) becomes infinite at the critical level. The critical layer noise ($i = 2$) is singular at $|n|\, \eta = 1$, but this is a consequence of the asymptotic approximations inherent in modulated wave theory, and the singularity is replaced by a boundary-layer structure in the full-wave theory of (BS 1982*a*). Finally the coefficients $\beta_2, \hat{\beta}_2$ (4.6) are given by

$$\beta_2, \hat{\beta}_2 = \pm \frac{2}{Nn} (|n|\, \eta - 1), \tag{4.12}$$

for $i = 1, 2$ respectively.

Next, consider the solutions $\{n, -\operatorname{sign} n, i, +\}$ which are given by

$$f' = (\operatorname{sign} n)\, \eta^{-1}, \ -n \tag{4.13}$$

for $i = 1, 2$ respectively. These solutions are analogous to those just discussed except that the group velocity is now positive. The first solution ($i = 1$) corresponds to a steady wave propagating upwards towards the critical level at $Z = 0$ if $Z < 0$, or away from the critical level in $Z > 0$. The second solution ($i = 2$) corresponds to upward-propagating critical-layer noise.

Now we turn to the general case and seek solutions $\{n, s, i, +\}$ with positive group velocity. From (4.3) these are given by

$$2\eta f' = -(n\eta + s) \pm \{(n\eta + s)^2 + 4\eta |n|\}^{\frac{1}{2}} \tag{4.14}$$

for $i = 1, 2$ respectively. For $s(\operatorname{sign} n) < -1$, both branches are defined for all η, except possibly $\eta = 0$. For $s > 0$ the branch $i = 1$ is regular at $\eta = O(f' \approx |n|\, s^{-1})$, while the branch $i = 2$ is singular ($f' \approx -s\eta^{-1}$); for $s < 0$ the branch $i = 1$ is singular ($f' \approx -s\eta^{-1}$), while the branch $i = 2$ is regular ($f' \approx |n|\, s^{-1}$). As $\eta \to \infty$, $f' \sim \eta^{-1}$ for $i = 1$ and $f' \sim -n - (s+1)\, \eta^{-1}$ for $i = 2$, when $n > 0$; if $n < 0$, $f' \sim -n - (s-1)\, \eta^{-1}$ for $i = 1$ and $f' \sim -\eta^{-1}$ for $i = 2$. Comparing these results with (4.13) we can interpret the branch for which $f' \sim (\operatorname{sign} n)\, \eta^{-1}$ as $\eta \to \infty$ as an upwardly propagating steady wave, and the branch for which $f' \sim -n$ as an upwardly propagating critical-layer noise. Similar considerations apply when $\eta \to -\infty$. The trajectories are found from (4.5) and are given by

$$T(f' + n) = \text{constant}. \tag{4.15}$$

Both branches are hyperbolae in the (Z, T)-plane and, depending on the sign of the constant in (4.15) and sign n, correspond either to a wave that propagates to $Z \to \infty$ in finite time, or to a finite value of Z as $T \to \infty$. The wave action equation (4.6a) can be integrated along each trajectory and we get

$$J = \text{constant} \times f''(f' + n)^{-2}. \tag{4.16}$$

Finally the coefficients $\beta_2, \hat{\beta}_2$ (4.6) are given by

$$\beta_2, \hat{\beta}_2 = \mp \frac{2}{N|n|}\{(n\eta + s)^2 + 4\eta|n|\}^{\frac{1}{2}} \tag{4.17}$$

for $i = 1, 2$ respectively.

For $s(\text{sign}\, n) \geqslant 0$ both branches are defined only in $\eta \geqslant \eta_1$ and $\eta \leqslant \eta_2$ where

$$|n|\,\eta_{1,2} = -\{2 + s(\text{sign}\, n)\} \pm 2\{1 + s(\text{sign}\, n)\}^{\frac{1}{2}}. \tag{4.18}$$

The two branches are equal at the turning points $\eta_{1,2}$ and we can regard the two branches as forming a single wave, one defined for $\eta \geqslant \eta_1$ and the other for $\eta \leqslant \eta_2$. Note that $\eta_2 < \eta_1 \leqslant 0$ and $\eta_1 = 0$ only if $s = 0$. The behaviour as $\eta \to 0$ or $\eta \to \pm\infty$ is the same as that described in the previous paragraph. The wave trajectories are again given by (4.15), and we note that since $\{(n\eta + s)^2 + 4\eta|n|\}^{\frac{1}{2}}$ vanishes at the turning points $\eta_{1,2}$, these mark a transition between branches. The trajectories are again hyperbolae in the (Z, T)-plane, and correspond either to a wave propagating to $Z \to \infty$ in finite time, possibly passing through a turning point, or to a wave that propagates to a finite value of Z as $T \to \infty$ again possibly passing through a turning point. The wave action is again given by (4.16) with $\beta_2, \hat{\beta}_2$ given by (4.17).

For the solutions $\{n, s, i, -\}$ with negative group velocity, we find that

$$2\eta f' = -(n\eta + s) \pm \{(n\eta + s)^2 - 4\eta|n|\}^{\frac{1}{2}}. \tag{4.19}$$

This can be analysed in a similar way to (4.14). We shall not give details except to observe that the transformation $\eta \to -\eta, n \to n, s \to -s$ and an exchange of branches takes (4.19) into (4.14).

4.2. *Resonant interactions*

With these preliminaries we now turn to an examination of wave interactions. Primary resonant triads (3.10) were discussed in detail in (G), and secondary resonant triads (3.17) can be discussed in a similar way. First let us note that for a primary resonant triad given by the waves $\{n_j, s_j, i, \pm\}$ for $j = 0, 1, 2$ the resonance conditions (3.10) imply that

$$n_0 + n_1 + n_2 = 0, \quad s_0 + s_1 + s_2 = 0, \tag{4.20a}$$

and

$$\sum_{j=0}^{2} f_j'(\eta) = 0, \tag{4.20b}$$

where of course each f_j' corresponds to the wave $\{n_j, s_j, i, \pm\}$. Using (4.14) and (4.20a) it can be shown that (4.20b) reduces to

$$\sum_{j=0}^{2} \pm\{(n_j\eta + s_j)^2 \pm 4\eta|n_j|\}^{\frac{1}{2}} = 0. \tag{4.21}$$

For a given pair of waves $j = 0, 1$ equations (4.20a) determine n_2, s_2 and then (4.20b) or (4.21) determines the values of η where a resonance can occur. It can be shown that (4.21) can be reduced to a quadratic equation in η, and when this has real solutions there are two possible resonance sites.

Next, for a secondary resonant triad, the resonance conditions (3.17) imply that

$$2n_0+n_1+n_2 = 0, \quad 2s_0+s_1+s_2 = 0 \tag{4.22a}$$

and

$$2f_0+f_1+f_2 = 0. \tag{4.22b}$$

Using (4.14) and (4.22*a*) it can be shown that (4.22*b*) reduces to

$$\pm 2\{(n_0\eta+s_0)^2 \pm 4\eta|n_0|\}^{\frac{1}{2}} + \sum_{j=1}^{2} \pm\{(n_j\eta+s_0)^2 \pm 4\eta|n_j|\}^{\frac{1}{2}} = 0. \tag{4.23}$$

As for (4.21) this can be reduced to a quadratic equation. Like the primary resonant triad, for a given pair of waves $j = 0, 1$ equations (4.22*a*) determine n_2, s_2 and then (4.22*b*) or (4.23) determines the values of η where a resonance can occur.

Here, we shall not explore the implications of the resonance conditions (4.20*a*, *b*) and (4.22*a*, *b*) in the general case. Instead we shall examine the sequence of interactions generated initially by the interaction of a steady wave and critical-layer noise, both approaching the critical level from above. Our purpose here is to model the scenario described by (BS) for the transient behaviour near a critical level. Importantly, both the incoming steady wave and the incoming critical-layer noise are generated by the same harmonic source, and hence we must put $|n_0| = |n_1|$ where the two waves are denoted by $j = 0, 1$ respectively. Hence we describe the incoming steady wave by $\{1, 1, 1, -\}$ with $f_0' = -\eta^{-1}$, and the incoming critical-layer noise by $\{-1, -1, 2, -\}$ with $f_1' = 1$ (see 4.8), i.e. $n_0 = 1$, $s_0 = 1$ and $n_1 = -1$, $s_1 = -1$. The respective amplitudes are given by (see the discussion following (4.11))

$$A_0 = C_0 \frac{T^{\frac{1}{2}}}{\eta^{\frac{1}{2}}} H(\eta-1), \quad A_1 = C_1 T^{\frac{1}{2}}(\eta-1)^{-1}, \tag{4.24}$$

where C_0, C_1 are constants, and we recall that η is given by (4.2*b*) so that $(T/\eta)^{\frac{1}{2}}$ is proportional to $Z^{-\frac{1}{2}}$.

It is readily shown that no primary resonant triads are possible. Note that the incoming critical-layer noise could also be described by its complex conjugate $\{1, 1, 2, -\}$ with $f_1' = -1$ and $n_1 = 1, s_1 = 1$. But it can again be shown that no primary resonant triads are possible in $\eta > 0$, which is where the incoming waves are specified. However, a secondary resonant triad is possible with $n_2 = -1$ and $s_2 = -1$, corresponding to the wave $\{-1, -1, i, \pm\}$. If the $j = 2$ wave is $\{-1, -1, i, -\}$ with a negative group velocity, then $f_2' = \eta^{-1}$ for $i = 1$ and $f_2' = 1$ for $i = 2$ (see (4.8)). But these are just the incoming steady wave and critical-layer noise respectively, and the only possible resonance at $\eta = 1$ is degenerate. Hence we must consider the $j = 2$ wave to be $\{-1, -1, i, +\}$ with a positive group velocity, where

$$2\eta f_2' = \eta+1 \pm \{\eta^2+6\eta+1\}^{\frac{1}{2}}. \tag{4.25}$$

Both branches are defined in $\eta > 0$ (in fact for $\eta > -3+\surd 8$). For the branch $i = 1$, $f_2' \sim 1+2\eta^{-1}$ as $\eta \to \infty$, while for $i = 2, f_2' \sim -\eta^{-1}$ as $\eta \to \infty$, corresponding respectively to critical-layer noise or a steady wave as $\eta \to \infty$. The possible resonance sites are found from (4.22*b*) or (4.23) and are given by $\eta = \frac{1}{2}(3 \pm \surd 5)$. Only $\eta = \frac{1}{2}(3+\surd 5)$ lies in $\eta > 1$ and is relevant since the incoming steady wave is only defined in $\eta > 1$. Further this resonance exists only for the branch $i = 2$. To summarize, the incoming steady wave and critical-layer noise undergo a secondary resonance at $\eta = \eta_0 = \frac{1}{2}(3+\surd 5)$ to produce the wave $\{-1, -1, 2, +\}$ which propagates upwards away from the critical level, and as $\eta \to \infty$ has the appearance of an upwardly propagating steady wave. This agrees with the results of (BS) who interpreted this as a mechanism for wave reflection from the critical level.

To this point there is no distinction between whether this secondary resonant triad is a second harmonic interaction (case (ii) in §3), or a bound harmonic interaction (case (iii) in §3). However, this distinction emerges when we calculate the amplitude A_2 at the resonance site, which is given by (3.21) for case (ii) or (3.28) in case (iii). Let us first consider (ii), the second harmonic interaction. Here we use (3.16*b*) to find $\gamma^{(2)}$, (4.17) for $\hat{\beta}_2$, (4.24) for A_0 and A_1, while $\hat{S}$ is given by

$$\hat{S} = N(2f_0''' + f_1''' + f_2''')_{\eta=\eta_0}. \tag{4.26}$$

This last expression is a consequence of (3.18) where we put $\hat{R} = (\eta - \eta_0)$ and use (3.15*a*) for $\hat{\chi}$. We find that $\hat{S} = 2\sqrt{5}N/3\nu_0^3$ where $\nu_0 = \frac{1}{2}(1+\sqrt{5})$ and $\eta_0 = \nu_0^2$. Next we calculate $\hat{\beta}_2$ from (4.17) and get $\hat{\beta}_2 = 6\omega_0/N$. It remains to calculate $\gamma^{(2)}$ from (3.16*b*). This is a lengthy calculation whose details we omit. With A_0 and A_1 given by (4.24) we can now deduce from (3.21) that, at the resonance site $\eta = \eta_0$, the amplitude $A_2^{(\mathrm{ii})}$ for this case (ii) is

$$A_2^{(\mathrm{ii})}(\eta = \eta_0) = -C_0^{*2} C_1^* \frac{K^2 T^{\frac{7}{2}}}{4N^{\frac{1}{2}}} \frac{(3\pi)^{\frac{1}{2}}}{5^{\frac{1}{4}}} \frac{(14 - 9\nu_0)}{\nu^{\frac{5}{2}}} \exp\left(-\frac{\mathrm{i}\pi}{4}\right). \tag{4.27}$$

Finally, we use the wave action equation (4.6*a*) with solution (4.16) to determine the amplitude $A_2^{(\mathrm{ii})}$ for values of η away from the resonance site $\eta = \eta_0$. The result is

$$A_2^{(\mathrm{ii})}/A_2^{(\mathrm{ii})}(\eta = \eta_0) = \left\{\frac{f_2''(f_2'-1)^6}{f_2'}\right\}^{\frac{1}{4}} \Big/ \left\{\frac{f_2''(f_2'-1)^6}{f_2'}\right\}^{\frac{1}{4}}_{(\eta=\eta_0)}, \tag{4.28}$$

where $f_2'(\eta)$ is defined by (4.25) with the $-$ sign. Finally as $\eta \to \infty$ we obtain

$$A_2^{(\mathrm{ii})} \sim R^{(\mathrm{ii})} C_0^* \frac{T^{\frac{1}{2}}}{\eta^{\frac{1}{2}}}, \tag{4.29a}$$

where
$$R^{(\mathrm{ii})} = -C_0^* C_1^* \frac{K^2 T^3}{N^{\frac{1}{2}}} \left\{\frac{3}{16} \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{4}}} \frac{(5 - 4\omega_0)}{(2\omega_0)^{\frac{1}{2}}} \exp\left(-\frac{\mathrm{i}\pi}{4}\right)\right\}. \tag{4.29b}$$

Comparing (4.29*a*) with (4.24) for the incident wave amplitude A_0, we see that $R^{(\mathrm{ii})}$ can be interpreted as a reflection coefficient due to this resonance mechanism. This definition is essentially the same as that used by (BS) and although they did not calculate the equivalent of $R^{(\mathrm{ii})}$ explicitly, the fact that $R^{(\mathrm{ii})}$ is proportional to T^3 is consistent with their analysis.

Next consider case (iii), the bound harmonic interaction. The calculation is similar to that for case (ii) described above, but now A_2 at the resonance site is given by (3.28). Here we use (3.26*b*) to calculate $\gamma^{(0)}$ while $\hat{\beta}_2$, A_0, A_1 and $\hat{S}$ are given by the same expressions as for case (ii). After a lengthy calculation we find that, at the resonance site $\eta = \eta_0$, the amplitude $A_2^{(\mathrm{iii})}$ for this case (iii) is

$$A_2^{(\mathrm{iii})}(\eta = \eta_0) = C_0^{*2} C_1^* \frac{K^2 T^{\frac{7}{2}} N^{\frac{1}{2}}}{3} \frac{(3\pi)^{\frac{1}{2}}}{5^{\frac{1}{4}}} \frac{1}{\nu_0^{\frac{7}{2}}} \exp\left(\frac{\mathrm{i}\pi}{4}\right). \tag{4.30}$$

Again we use the wave action equation (4.6*a*) with solution (4.16) to determine the amplitude $A_2^{(\mathrm{iii})}$ for values of η away from the resonance site $\eta = \eta_0$. The result is the same as (4.28) with the index (iii) now replacing (ii). Finally as $\eta \to 0$ we obtain

$$A_2^{(\mathrm{iii})} \sim R^{(\mathrm{iii})} \frac{C_0^* T^{\frac{1}{2}}}{\eta^{\frac{1}{2}}},$$

where
$$R^{(iii)} = C_0^* C_1^* K^2 T^3 N^{\frac{1}{2}} \left\{ \frac{1}{8} \frac{\pi^{\frac{1}{2}}}{5^{\frac{1}{4}}} \frac{1}{(2\omega_0)^{\frac{1}{2}}} \exp\left(\frac{i\pi}{4}\right) \right\}. \tag{4.31}$$

Remarkably, this expression for $R^{(iii)}$ agrees exactly with the corresponding reflection coefficient calculated by (BS). The calculation of $R^{(ii)}$ and $R^{(iii)}$ are the main results of this paper. But in comparing the respective magnitudes it must be recalled that for the second harmonic interaction (ii) the amplitude parameter $\delta = \alpha_0^2 \alpha_1 \epsilon^{\frac{1}{2}}$ whereas for the bound harmonic interaction (ii) $\delta = \alpha_0^2 \alpha_1 \epsilon^{-\frac{1}{2}}$ and hence the reflected wave for the bound interaction is larger by a factor of ϵ^{-1} than for the reflected wave for the second-harmonic interaction.

Next we consider the case when the incoming steady wave is again described by $\{1, 1, 1, -\}$ but the incoming critical-layer noise is described by its complex conjugate $\{1, 1, 2, -\}$. For a secondary resonant triad to be possible we now have $n_3 = -3$ and $s_3 = -3$. But it can be shown that no resonances are possible in $\eta > 0$, and since the incoming steady wave is only defined in $\eta > 1$, we see that this need not be discussed further.

Another possibility is to interchange the roles of the incoming steady wave and incoming critical-layer noise, so that $j = 0$ is $\{-1, -1, 2, -\}$ with $f_0' = 1$ and $n_0 = -1$, $s_0 = -1$ (incoming critical-layer noise), while $j = 1$ is $\{1, 1, 1, -\}$ with $f_1' = -\eta^{-1}$ and $n_1 = 1, s_1 = 1$ (incoming steady wave). Of course this makes no difference for primary resonant triads for which the system is symmetric with respect to $j = 0, 1$. But for a secondary resonant triad it is now the critical-layer noise that has the distinguished role. A secondary resonant triad is possible with $n_2 = 1, s_2 = 1$ corresponding to the wave $\{1, 1, i, \pm\}$. If this $j = 2$ wave is $\{1, 1, i, -\}$ with negative group velocity then $f_2' = -\eta^{-1}$ for $i = 1$ and $f_2' = -1$ for $i = 2$, and the system is degenerate. Hence we must choose the $j = 2$ wave to be $\{1, 1, i, +\}$ with positive group velocity and
$$2\eta f_2' = -(\eta + 1) \pm \{\eta^2 + 6\eta + 1\}^{\frac{1}{2}}. \tag{4.32}$$

Note that these are precisely the same wave possibilities considered previously (see 4.24)). The only possible resonance is again at $\eta = \eta_0 = \frac{1}{2}(3 + \surd 5)$, and only exists for the branch $i = 2$. As $\eta \to \infty$ this branch behaves as critical-layer noise since then $f_2' \sim -1$. To summarize, the incoming critical-layer noise and steady wave undergo a secondary resonance at $\eta = \eta_0 = \frac{1}{2}(3 + \surd 5)$ to produce the wave $\{1, 1, 2, +\}$ which propagates upwards away from the critical level and as $\eta \to \infty$ has the appearance of upwardly propagating critical-layer noise. It would now be possible to calculate the corresponding amplitudes A_2 for cases (ii) and (iii) in the same manner as described above. However, we shall not do this since this resonance does not produce wave reflection in the sense defined by (BS).

(BS) go on to consider higher-order interactions than those described here in §3, and in particular calculate the coefficient of the second harmonic of the reflected wave which is produced by a resonance between a bound harmonic with phase $(2\theta_0 + \theta_1)$ and the first harmonic of the mode with phase θ_0. They also show that even higher-order interactions can produce a transmitted wave, but do not calculate the transmission coefficient explicitly.

Here, however, we have chosen to confine our attention to the resonances (i), (ii) and (iii) described in §3. So far we have considered the resonant interactions between an incoming steady wave and incoming critical layer noise and found that a third wave is generated at the resonance site $\eta = \eta_0 = \frac{1}{2}(3 + \surd 5)$ and propagates upwards. But now we have the possibility to consider further resonant interactions between either an incoming steady wave or incoming critical-layer noise ($j = 0$ or 1) and the newly generated wave ($j = 1$ or 0) with phase function $f(\eta)$ given by $2\eta f' = \eta + 1 - \{\eta^2 + 6n + 1\}^{\frac{1}{2}}$ (see 4.25) and (4.42)). Considering both primary and secondary resonant triads, and taking into account all possibilities we find that no such further resonances are possible in $\eta > \eta_0$ which is where the newly generated wave is defined. Hence in the context of this study the calculation of possible resonances generated within the critical layer is completed.

5. Discussion

In this paper we have developed a general theory for secondary resonant triad wave interactions in a stratified shear flow, thus extending the analysis of (G) which considered primary resonant triads. The main purpose of this theory was to consider the resonant interaction between waves near a critical level, with the aim of showing that the study by (BS) of nonlinear processes near a critical level can be interpreted by the mechanisms described in this paper. As in (BS) we considered two waves approaching a critical level, these being in their terminology a steady wave and critical-layer noise. Then we showed that a secondary resonance, defined by (3.17), in which two harmonic components of the steady wave interact with a single harmonic component of the critical-level noise, produces a reflected wave. This result agrees with (BS), and indeed our calculation of the reflection coefficient (4.41) agrees with theirs. The (BS) result was obtained for a two-dimensional flow configuration, with linear background shear and uniform density stratification, and utilizing the hydrostatic approximation. Even so, their analysis involved very complicated and technically difficult asymptotic and perturbation methods, which do not reveal easily the underlying wave resonance mechanisms. Even though our reanalysis of the transient critical layer in §4 uses the same specializations as (BS), we contend that placing the analysis in the context of a general theory of wave resonances reveals more clearly the central role of wave resonances, and indicates how the transient critical layer might be understood in more general circumstances.

Since here we have not gone beyond a secondary resonance, we are unable to reproduce the higher-order calculations of (BS). But it is straightforward to describe the possible resonance conditions. Thus let $j = 0$ denote the steady wave with phase θ_0, and let $j = 1$ denote the critical-layer noise with phase θ_1, as in §4. Then an interaction between P components of $j = 0$ and Q components of $j = 1$ will produce the wave $j = 2$ with phase θ_0, provided that the following resonance conditions are satisfied:

$$P\omega_0 + Q\omega_1 + \omega_2 = 0, \quad P\boldsymbol{\kappa}_0 + Q\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 = 0. \tag{5.1}$$

For the phases θ given by (4.2a) these imply that

$$Pn_0 + Qn_1 + n_2 = 0, \quad Ps_0 + Qs_1 + s_2 = 0, \tag{5.2a}$$

and

$$Pf_0' + Qf_1' + f_2' = 0. \tag{5.2b}$$

Further, since here $n_0 = 1, n_1 = -1, s_0 = 1, s_1 = -1$ and $f_0' = -\eta^{-1}, f_1' = 1$, these reduce to

$$n_1 = Q - P, \quad s_2 = Q - P, \tag{5.3a}$$

and

$$f_2' = \frac{P}{\eta} - Q. \tag{5.3b}$$

Thus (5.3a) determines that the wave produced by the resonance is $\{n_2, s_2, i, +\}$ which propagates vertically upwards, since with $n_2 = s_2$ as here the analysis of §4 shows that

$\{n_2, s_2, i, -\}$ consists of a steady wave ($i = 1$) and critical-layer noise ($i = 2$) and the resonance is degenerate. From (4.14) it is seen that the $j = 2$ wave is defined in $\eta \geqslant \eta_1$ and $\eta \leqslant \eta_2$ where (see 4.18))

$$|n_2|\,\eta_{1,2} = -\{2 + |n_2|\} \pm 2\{1 + |n_2|\}^{\frac{1}{2}}. \tag{5.4}$$

Since here $\eta_{1,2} < 0$ it follows that the $j = 2$ wave always exist in the region $\eta > 1$ where the incoming steady wave is defined. Further as $\eta \to \infty$ the branch $i = 1$ has $f_2' \sim \eta^{-1}$ for $n_2 > 0$ and can be interpreted as a steady wave, while the branch $i = 2$ has $f_2' \sim -n_2$ and can be interpreted as critical-layer noise. When $n_2 < 0$, the branch $i = 1$ has $f_2' \sim -n_2$ (critical-layer noise) and the branch $i = 2$ has $f_2' \sim -\eta$ (steady wave). Finally (5.3b) determines the possible resonance sites. If these lie in $\eta > 1$, then the above analysis shows that the $j = 2$ wave produced by this (P, Q) resonance propagates upwards and as $\eta \to \infty$ is either a steady wave or critical-layer noise. The scenario is essentially the same as the (2, 1) or (1, 2) resonance analysed in §4.

For instance, if we put $P = 3, Q = 1$, then $n_2 = s_2 = -2$ and the resonantly generated wave is a second harmonic (see (4.2a) and note that $k_2 = -2K$). The resonance site is $\eta = 2 + \surd 3$, and the resonantly generated wave has $i = 2$, and hence behaves as a steady wave as $\eta \to \infty$. This agrees with the analysis of (BS) [illegible] go on to calculate the amplitude of this wave, and interpret it as the second harmonic of the reflected wave, whose first harmonic was described in §4. Note that to get a first harmonic for the reflected wave, we must choose $n_2 = s_2 = \pm 1$, so that $Q - P = \pm 1$, and indeed the analysis of §4 had $P = 2, Q = 1$, although clearly there are many other possibilities involving higher-order interactions.

Finally, we note that the scenario described above in which the waves generating the resonant interaction are the incoming steady wave and critical-layer noise, cannot produce a transmitted wave. As shown by (BS), for this to occur we must consider resonant interactions between either the incoming steady wave, or the incoming critical-layer noise, and one of the resonantly generated waves as described above. Our analysis at the end of §4 shows that no transmitted wave can be generated by primary or secondary resonances of this kind, but (BS) show that higher-order resonances will eventually produce a transmitted wave.

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